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Lyapunov Optimal Saturated Control for Nonlinear Systems

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A generalized feedback control law design methodology is presented that applies to systems under control saturation constraints. Lyapunov stability theory is used to develop stable saturated control laws that can be augmented to any unsaturated control law that transitions continuously at a touch point on the saturation boundary. The time derivative of the Lyapunov function, an error energy measure, is used as the performance index, which provides a measure that is invariant to the system dynamics. Lyapunov stability theory is used constructively to establish stability characteristics of the closed-loop dynamics. Lyapunov optimal control laws are developed by minimizing the performance index over the set of admissible controls, which is equivalent to forcing the error energy rate to be as negative as possible.

Introduction

THE slewing of precision pointing spacecraft with reaction wheels has produced a need to address the stability of systems under saturated control. If a slew maneuver is performed that saturates one or more of the reaction wheels, does the spacecraft remain stable? Should the feedback gains be scaled back to keep all of the controls in the unsaturated region, or is it more beneficial to let some controls saturate and others operate unsaturated? These questions have led to the development of nonlinear optimal feedback control systems that are designed by Lyapunov's direct method and remain stable under saturated conditions.

The concept of optimal feedback controllers that are designed with Lyapunov stability theory can be found as far back in the literature as a homework problem in Ref. 1, which originated in issues raised by Ref. 2. Kalman and Bertram[2] introduced the idea of designing an optimal controller for a linear system that has saturation constraints. In this case, the controller design is performed for the entire state-space region, i.e., both saturated and unsaturated. More present day applications include developments in Refs. 3 and 4, where the concept of Lyapunov optimal is utilized for feedback controller design. A control law is Lyapunov optimal if it minimizes the first time derivative of the Lyapunov function over a space of admissible controls. More generally, a set of feedback gains is optimized by minimizing the tracking error of the feedback controller while tracking a specific reference maneuver.

In this paper, we employ the concept of Lyapunov optimality to design stable, saturated controllers for nonlinear systems. Sensor and actuator dynamics have been neglected to focus on the effects of control saturation. The Lyapunov function is the total error energy, which for most mechanical systems is equivalent to an appropriate Hamiltonian function.[5] The performance index is the first time derivative of the Lyapunov function, which is the instantaneous work rate and is a kinematic relationship independent of the system dynamics.[5] This truth is fundamental because it means the structure of the controller depends only on the kinematic model, and so the same control law stabilizes a large class of dynamical models. This means a high degree of robustness is implicit in this approach. A key region of interest is all of the state space where the controls are saturated. As a result of developments herein, the unsaturated control space is only considered when the saturated controller approaches the saturation boundary, which may be a touch point. Upon passing through the saturation boundary, an unbounded optimal feedback controller can be employed in the unsaturated region. It is usually desirable, especially for flexible structures, that control transitions be continuous.

General Controller Design

Lyapunov's direct method is attractive to design globally asymptotically stable, nonlinear feedback controllers. The Lyapunov function is typically chosen as the total error energy of the system

$$U = T + V \tag{1}$$

where T is the kinetic energy and V is the potential energy. The implicit reference state in Eq. (1) is the target state. U is typically positive definite, but when it is not, an additive fictitious energy function[3,4] equivalent to a position feedback loop with to-be-designed feedback gains can be used to modify U appropriately. Because most mechanical systems are natural systems, the Hamiltonian specializes for this case to the total system energy, which motivates the alternative use of the Hamiltonian as a more general Lyapunov function candidate.[5] The reference state in Eq. (1) is the target state; thus, U can be interpreted as the error energy of the system.

The first time derivative of the Lyapunov function in Eq. (1) is the instantaneous work rate

$$\dot{U} = \dot{H}(\boldsymbol{p}, \boldsymbol{q}) = \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} Q_i \tag{2}$$

where H is the Hamiltonian, $\boldsymbol{q} = \boldsymbol{q}(t)$ is the n-dimensional generalized coordinate vector, $\boldsymbol{p}$ is the n-dimensional generalized momentum vector, $\boldsymbol{Q}$ is the generalized force vector, and $\dot{(\,)} \equiv \mathrm{d}/\mathrm{d}t(\,)$. To be more specific,[6] L is defined as the system Lagrangian, where

$$L = T(\boldsymbol{q}, \dot{\boldsymbol{q}}) - V(\boldsymbol{q}) \tag{3}$$

with the classical definitions

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{4}$$

and

$$H(\boldsymbol{q}, \boldsymbol{p}) = \sum_{i=1}^{n} p_i \dot{q}_i - L(\boldsymbol{q}, \dot{\boldsymbol{q}}) \tag{5}$$

leading to Hamilton's canonical equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{6}$$

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and

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} + Q_i \tag{7}$$

The combination of Eqs. (2–7) produces the power (work/energy) equation[7–9]

$$\dot{U} = \sum_{i=1}^{n} \dot{q}_i Q_i \equiv \sum_{i=1}^{M} L_i \cdot \omega_i + \sum_{i=1}^{N} F_i \cdot \dot{R}_i \tag{8}$$

which is independent of the system dynamics. Equation (8) is a kinematic quantity that applies to any system and is an ideal performance index for this problem. In Eq. (8), $\{F_1, \ldots, F_N\}$ and $\{L_1, \ldots, L_M\}$ denote a set of forces and moments acting on a mechanical system. $\dot{R}_i$ denotes the inertial linear velocity of the point where F_i is applied. The component ω_i denotes the inertial angular velocity of the point where L_i is applied. Note that Eq. (8) can be written immediately without further reference to the dynamical modeling assumptions and, therefore, holds for an infinity of systems. It is implicitly necessary, however, that the actual system must be controllable and the actual Hamiltonian must be positive definite with respect to departures from the target state. Otherwise, it is necessary to establish sufficient insight to modify U and/or the number of control inputs.[3]

The goal of the controller design process is to choose a control law, i.e., select the equation form of the generalized forces, from an admissible set that will stabilize the system in an optimal fashion, i.e., make $\dot{U}$ as negative as possible. For saturated controls, the classical stabilizing controller takes the form

$$Q_i = -Q_{i_{\max}} \operatorname{sgn}(\dot{q}_i) \tag{9}$$

which is Lyapunov optimal for the performance index

$$J = \dot{U} = \sum_{i=1}^{n} \dot{q}_i Q_i \tag{10}$$

The control law is optimal in a sense analogous to Pontryagin's principle for optimal control because the controls are selected from an admissible set $|Q_i| \le Q_{i_{\max}}$ such that the instantaneous work rate is minimized at every point in time. Note that mathematical difficulties and practical system performance issues arise if this controller is implemented directly for most systems.[2] The discontinuity at the origin must typically be replaced with a region of unsaturated control to avoid chattering near $\dot{q}_i = 0$. This unsaturated controller can either approximate the discontinuity or be some other stable/optimal feedback controller that transitions from the saturated controller on the saturation boundary. We restrict attention to control laws that transition continuously at the saturation boundary. The obvious choice is to augment Eq. (9) with a linear controller of the type

$$Q_i = \begin{cases} -K_i \dot{q}_i & \text{for} \quad |K_i \dot{q}_i| \le Q_{i_{\max}} \\ -Q_{i_{\max}} \operatorname{sgn}(\dot{q}_i) & \text{for} \quad |K_i \dot{q}_i| > Q_{i_{\max}} \end{cases} \tag{11}$$

where $K_i > 0$ is a chosen feedback gain. This control continuously transitions across the saturation boundary and eliminates chattering. Note that Eq. (11) allows some elements of the control vector to become saturated, whereas others are still in the unsaturated range. This differs from conventional gain scheduling and deadband methods, which typically reduce the feedback gains to keep all controls in the unsaturated range.

Two examples are solved in this section to demonstrate the generality of this feedback controller design process; subsequent examples of increasing dimensionality and generality are used to further illustrate the ideas.

Example 1: Duffing Oscillator

The first example is the design of a control law for a single-degree-of-freedom nonlinear oscillator. Assuming $m, c, k, k_N > 0$, the equation of motion is

$$m\ddot{x} + c\dot{x} + kx + k_N x^3 = u \tag{12}$$

and the Lyapunov function (the system Hamiltonian of the unforced and undamped system) is

$$U = \tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{2} kx^2 + \tfrac{1}{4} k_N x^4 \tag{13}$$

The performance index is the time derivative of Eq. (13) and can be written immediately from Eq. (8) as

$$J = \dot{U} = \dot{x} Q = \dot{x}(-c\dot{x} + u) \tag{14}$$

For bounded control $|u| \le u_{\max}$, the performance index J is minimized by the feedback controller

$$u = -u_{\max} \operatorname{sgn}(\dot{x}) \tag{15}$$

Using this control law, $\dot{U}$ is reduced to the energy dissipation rate

$$\dot{U}(x, \dot{x}) = -c\dot{x}^2 - u_{\max}\dot{x} \cdot \operatorname{sgn}(\dot{x}) \tag{16}$$

Note that an arbitrary, unknown, positive definite potential energy function $\Delta V(x)$ could be added to U in Eq. (13) and the negative gradient of ΔV to Eq. (12), and exactly the same result would be obtained for Eqs. (14) and (15). Thus, the structure of the control law and the stability guarantee is invariant with respect to a large family of modeling assumptions.

Inasmuch as $\dot{U}(x, \dot{x})$ of Eq. (16) is negative semidefinite, it can only be concluded at this point that the system is stable; thus, x and $\dot{x}$ will remain bounded. Because the control u is bounded by definition, Eq. (12) shows that $\ddot{x}$ will also be bounded. To prove asymptotic stability, the higher derivatives of U must be investigated. A sufficient condition to guarantee asymptotic stability is that the first nonzero higher-order derivative of U, evaluated on the set of states such that $\dot{U}$ is zero, must be of odd order and negative definite.[9] The only equilibrium point where $\dot{U}$ vanishes is $\dot{x} = 0$. The second derivative of U is

$$\frac{\mathrm{d}^2 U}{\mathrm{d}t^2} = -2c\dot{x}\ddot{x} - u_{\max}\ddot{x} \operatorname{sgn}(\dot{x}) \tag{17}$$

which is zero for all x when $\dot{x} = 0$. The third derivative of U is

$$\frac{\mathrm{d}^3 U}{\mathrm{d}t^3} = -2c\ddot{x}^2 - 2c\dot{x}\frac{\mathrm{d}^3 x}{\mathrm{d}t^3} - u_{\max}\frac{\mathrm{d}^3 x}{\mathrm{d}t^3} \operatorname{sgn}(\dot{x}) \tag{18}$$

Using Eq. (12), we find on the set where $\dot{x} = 0$ that

$$\left.\frac{\mathrm{d}^3 U}{\mathrm{d}t^3}\right|_{\dot{x}=0} = -2\frac{c}{m}\left(kx + k_N x^3\right)^2 \tag{19}$$

which is a negative definite function of x. Therefore, the saturated control law in Eq. (15) is globally asymptotically stabilizing.

Example 2: Rigid Body Detumbling

The second example is detumbling a rigid body spacecraft to zero angular velocity at an unspecified orientation. Although control torque saturation is considered in this rigid body example, control momentum saturation has not been addressed. The equations of motion are

$$I\dot{\boldsymbol{\omega}} + [\tilde{\omega}]I\boldsymbol{\omega} = \boldsymbol{u} \tag{20}$$

where I is the positive definite matrix of principal moments of inertia, $\boldsymbol{\omega}$ is the body-fixed angular rate vector instead of the generalized coordinate $\dot{\boldsymbol{q}}$, and $\boldsymbol{u}$ is the control torque vector. The tilde matrix $[\tilde{\omega}]$ is defined as

$$[\tilde{\omega}] = \begin{bmatrix} 0 & -\boldsymbol{\omega}_3 & \boldsymbol{\omega}_2 \\ \boldsymbol{\omega}_3 & 0 & -\boldsymbol{\omega}_1 \\ -\boldsymbol{\omega}_2 & \omega_1 & 0 \end{bmatrix} \tag{21}$$

Because the rigid body orientation is unspecified, we can consider $\boldsymbol{\omega}$ to be the state vector, and the system is of order three. The Lyapunov function is then the system kinetic energy

$$U = \tfrac{1}{2}\boldsymbol{\omega}^T I \boldsymbol{\omega} \tag{22}$$

and the time derivative of Eq. (22) is adopted as the performance index (energy rate)

$$J = \dot{U} = \boldsymbol{\omega}^T \boldsymbol{u} \tag{23}$$

We note Eq. (23) is simply the (kinematic) work/energy equation, which we have written immediately. In this case, formal differentiation of Eq. (22), substitution for $I\dot{\omega}$ from Eq. (20), requires we recognize or verify that the work rate of the gyroscopic term is zero, i.e., that $\boldsymbol{\omega}^T([\tilde{\omega}]I\boldsymbol{\omega}) = 0$. For more complicated dynamics, the use of Eq. (8) saves considerable algebra. Thus, it is not necessary to reinvent the work/energy equation for each special case; we know it already. Obviously, an infinite set of controls makes $\dot{U}$ negative definite in Eq. (23), but for bounded controls $\{|u_i| < u_{\max_i}\}$ the resulting Lyapunov optimal control law that minimizes Eq. (23) is clearly

$$\boldsymbol{u} = -A\,\mathrm{sgn}(\boldsymbol{\omega}) \tag{24}$$

where we use the notational compaction

$$\mathrm{sgn}(\boldsymbol{\omega}) = [\mathrm{sgn}(\omega_1),\ \mathrm{sgn}(\omega_2),\ \mathrm{sgn}(\omega_3)]^T \tag{25}$$

and A is a positive definite diagonal weight matrix:

$$A = \begin{bmatrix} u_{\max_1} & 0 & 0 \\ 0 & u_{\max_2} & 0 \\ 0 & 0 & u_{\max_3} \end{bmatrix} \tag{26}$$

This control law minimizes the performance index $\dot{U}$ to

$$J = \dot{U} = -\boldsymbol{\omega}^T A\,\mathrm{sgn}(\boldsymbol{\omega}) \tag{27}$$

which is negative definite. Therefore, $\boldsymbol{u}$ is a globally asymptotically stabilizing saturated control able to detumble a rigid body from any arbitrary rotation to rest.

As was pointed out, a direct implementation of this saturated control will cause chattering around $\boldsymbol{\omega} = 0$. As outlined in Eq. (11), the saturated control torque is augmented with the linear unsaturated controller of the type

$$u_i = \begin{cases} -k_i\omega_i & \text{for} \quad |k_i\omega_i| \le u_{\max_i} \\ -u_{\max_i}\,\mathrm{sgn}(\omega_i) & \text{for} \quad |k_i\omega_i| > u_{\max_i} \end{cases} \tag{28}$$

where k_i are chosen feedback gains.

Tracking Controller Design

To include the control problem for a slewing spacecraft, the design of tracking controllers that remain stable under saturated conditions must be considered. The formulation of the prior section is modified to accommodate tracking a reference motion $\boldsymbol{x}_r(t)$ by rewriting Eq. (1) as the total tracking error energy

$$U_T = \Delta T + \Delta V \tag{29}$$

In this case, the concept of Lyapunov optimality is difficult to define because tracking stability cannot typically be guaranteed during the intervals while the controls are saturated. Nevertheless, globally asymptotically, stable tracking controllers can often be achieved by generalizing the method used in the preceding section. A generalized work/energy equation that is equivalent to Eq. (8) is not possible because the position and/or attitude error tracking coordinates are measured in a noninertial reference frame. Also, consideration must be given to whether the prescribed trajectory is a feasible exact trajectory of the system. The following two examples, motivated by Refs. 5 and 7, demonstrate the tracking feedback controller design process.

Example 3: Duffing Oscillator, Trajectory Tracking

The nonlinear oscillator of Eq. (12) is discussed first. Let the tracking error Δx be given as

$$\Delta x = x - x_r \tag{30}$$

The Lyapunov function that is interpreted as the total tracking error energy is defined as

$$U_T = \tfrac{1}{2}m\Delta\dot{x}^2 + \tfrac{1}{2}k\Delta x^2 + \tfrac{1}{4}k_N\Delta x^4 \tag{31}$$

The reference trajectory $x_r(t)$ is any smooth differentiable function. The first time derivative of Eq. (31) is

$$\dot{U}_T = \Delta\dot{x}\left[m\Delta\ddot{x} + k\Delta x + k_N\Delta x^3\right] \tag{32}$$

which, making use of Eq. (12) and requiring Eq. (32) to be negative, leads to the following unsaturated feedforward/feedback controller:

$$u_{\mathrm{us}} = \left[m\ddot{x}_r + c\dot{x}_r + kx_r + k_N x_r^3\right] + 3k_N x x_r\Delta x - A\Delta\dot{x} \tag{33}$$

and for saturated control, to minimize $\dot{U}_T$ in Eq. (32), we find

$$u = u_{\max}\,\mathrm{sgn}(u_{\mathrm{us}}) \tag{34}$$

where A is a positive feedback gain. Using u_{us} from Eq. (33) in Eqs. (12) and (32), the performance index $\dot{U}_T$ becomes

$$\dot{U}_T = -(c + A)\Delta\dot{x}^2 \tag{35}$$

which is negative semidefinite. Therefore, the tracking errors Δx and $\Delta\dot{x}$ will be stable and bounded in the absence of model errors. Assuming the reference motion $x_r(t)$ is bounded, then the control torque u_{us} will also be bounded. To investigate asymptotic stability with respect to the reference trajectory using unsaturated control, let us investigate the higher derivatives of U_T. The only equilibrium point of $\dot{U}_T$ occurs where $\Delta\dot{x} = 0$. The second derivative of U_T is

$$\frac{\mathrm{d}^2U_T}{\mathrm{d}t^2} = -2(c + A)\Delta\dot{x}\Delta\ddot{x} \tag{36}$$

Because Δx, $\Delta\dot{x}$, and u_{us} are bounded, Eqs. (12) and (30) can be used to show that $\Delta\ddot{x}$ is bounded. Thus, the second derivative of U_T is zero at the equilibrium point. The third derivative of U_T is

$$\frac{\mathrm{d}^3U_T}{\mathrm{d}t^3} = -2(c + A)\Delta\ddot{x}^2 - 2(c + A)\Delta\dot{x}\frac{\mathrm{d}^3\Delta x}{\mathrm{d}t^3} \tag{37}$$

Using Eq. (12), this can be evaluated on the set of states for which $\Delta\dot{x}$ vanishes as

$$\left.\frac{\mathrm{d}^3U_T}{\mathrm{d}t^3}\right|_{\Delta\dot{x}=0} = -2\frac{c + A}{m}\left(k\Delta x + k_N\Delta x^3\right)^2 \tag{38}$$

which is negative definite for any tracking error Δx. Thus, the unsaturated feedforward/feedback controller u_{us} is globally asymptotically stabilizing in the absence of feedforward model errors.[9]

It is prudent to evaluate the stability boundaries of the saturated control law of Eq. (34). The first step is to neglect the nonlinear terms and rewrite Eq. (34) as

$$u = u_{\max}\,\mathrm{sgn}([m\ddot{x}_r + c\dot{x}_r + kx_r] - A\Delta\dot{x}) \tag{39}$$

which imposes the sufficient stability constraint of

$$|m\ddot{x}_r + c\dot{x}_r + kx_r| \le |u_{\max}| \tag{40}$$

In other words, the required force to track the reference maneuver cannot exceed the maximum control authority. In hindsight this is an obvious result, if one expects to track an arbitrary reference trajectory. The second step is to include the nonlinear terms of Eq. (34) that produce the following sufficient stability constraint condition:

$$\left|\left(m\ddot{x}_r + c\dot{x}_r + kx_r + k_N x_r^3\right) + 3k_N x x_r\Delta x\right| \le |u_{\max}| \tag{41}$$

Whereas the linear approximate result of Eq. (40) only depends on the absolute reference trajectory, Eq. (41) shows that the nonlinear stability boundary depends on the time history of the tracking error, as well as the reference trajectory. One way to interpret the stability constraint of Eq. (41) is that the rate of growth of the tracking error under saturated conditions must be limited to the remaining control authority after accounting for the reference trajectory requirements.

This may be illustrated by expressing the stability constraint of Eq. (41) as

$$|u_{\mathrm{us}} + A\Delta\dot{x}| \le |u_{\max}| \tag{42}$$

Employing the triangle inequality yields the sufficient condition

$$|A\Delta\dot{x}| \le |u_{\max}| - |u_{\mathrm{us}}| \tag{43}$$

which is a conservative stability constraint on $A\Delta\dot{x}$. Positive values of A cause the control to saturate earlier than if a control effort bound is simply placed on Eq. (33) for $A = 0$. However, under the saturated condition, the system is still stable so long as Eq. (41), or more conservatively Eq. (43), is satisfied. Because closed-loop stability is dependent on the reference maneuver and predicted dispersions off the reference maneuver, trajectory design is an iterative procedure.

Continuing with example 3, system stability and trajectory design issues will be further exemplified by simulation results using numerical values for this example. The mass m is 1 kg, the stiffness k is 5 N/m, the nonlinear stiffness k_N is 25 N/m^3, and the damping c is 0.1 Ns/m. The velocity feedback gain A is 100, and the maximum controller effort $u_{\max}$ is set at 30 N. The reference trajectory adopted is

$$x_r = 0.5 + 0.4\sin(2\pi t \times 1.5\ \mathrm{Hz}) \tag{44}$$

Notice that the maximum force required for the mass m to track x_r is larger than $u_{\max}$, saturating the actual applied force. Although the lack of compatibility between $u_{\max}$ and $x_r(t)$ is easily resolved by changing either, we consider this difficulty because we often face this situation in practice. To help illustrate the difference between control saturation and stability constraint violation, the time regions where the stability constraint of Eq. (41) is violated are grayed out in the following figures.

The tracking errors shown in Fig. 1 grow during the intervals where the stability constraint is violated. Likewise, there is a slight decrease in tracking error between these unstable regions, consistent with the theoretical asymptotic stability for the unsaturated regions.

Figure 2 shows the control force u. Although the control clearly saturates, the stability constraint is violated only during subintervals of the torque saturation regions. A plot of $\dot{U}_T$ is shown in Fig. 3. Recall that the stability constraint of Eq. (41) is derived from enforcing $\dot{U}_T$ to be negative. For this one-dimensional system, the region where the stability inequality Eq. (41) fails corresponds directly with the region of positive $\dot{U}_T$. For higher-dimensional systems, this type of stability condition will typically provide a much more conservative estimate of the stable region boundary than what the actual stability region boundary really is. This truth will be illustrated in the following example.

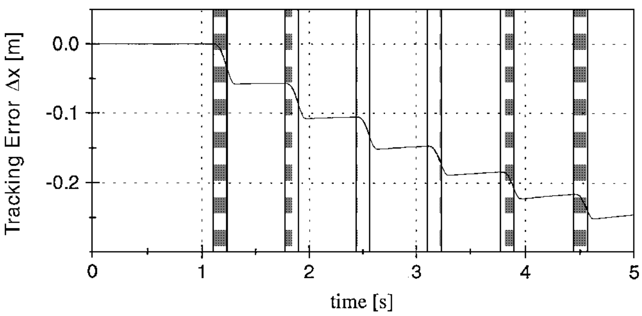

Fig. 1 Tracking error.

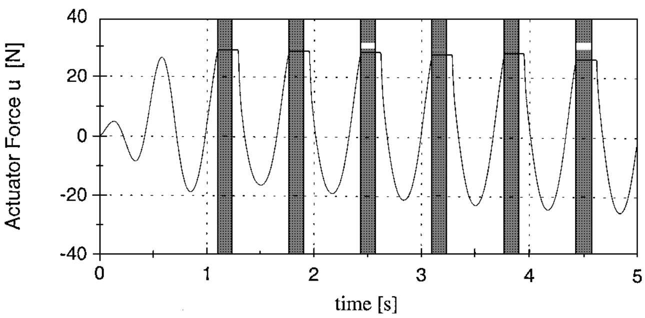

Fig. 2 Actuator force.

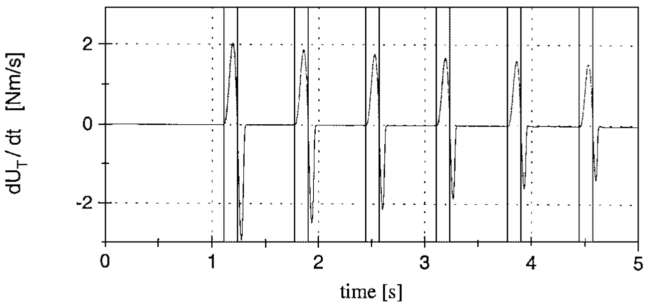

Fig. 3 Time derivative of U_T.

In practice, the reference maneuver can be designed with Eq. (41) in mind to allow an adequate margin for a finite Δx stability region. An important issue for practical applications is the dependence of this process on knowledge of the system dynamical model.

Example 4: Detumbling Rigid Body to Specific State

Another tracking example deals with detumbling a rigid body and requiring it to track a prescribed reference trajectory. The body orientation relative to an inertial frame N is given through the 3×3 orientation (direction cosine) matrix $[BN]$. The reference orientation is given by the orientation matrix $[RN]$. The relative orientation between the actual and the reference orientation is given at any instant by $[BR] = [BN][RN]^T$. The attitude tracking error is then described by the modified Rodrigues parameter[10–13] vector $\boldsymbol{\sigma}$, which minimally parameterizes the $[BR]$ matrix. Choosing $\boldsymbol{\sigma}$ to parameterize the attitude error is one of many possibilities, but as is evident subsequently, this leads to a very attractive control law. Among many other advantages, these three parameters are nonsingular for all possible ± 180-deg rotations and have near-linear kinematics for up to ± 90-deg rotations.[10,14,15] For tracking motions, we are virtually certain of always being in the near-linear range. This truth vastly expands the ranges of physical motions over which linear control theory (for gain design) is valid. The differential kinematic equation for $\boldsymbol{\sigma}$ is

$$\frac{\mathrm{d}\boldsymbol{\sigma}}{\mathrm{d}t} = \frac{1}{2}\left[I\left(\frac{1 - \boldsymbol{\sigma}^T\boldsymbol{\sigma}}{2}\right) + [\tilde{\sigma}] + \boldsymbol{\sigma}\boldsymbol{\sigma}^T\right]\delta\boldsymbol{\omega} \tag{45}$$

Let $\boldsymbol{\omega}$ be the actual body angular velocity written in the body-fixed coordinate system, and let $\boldsymbol{\omega}_r$ be the reference body angular velocity written in the reference attitude coordinate system. Then the error in body angular velocity is

$$\delta\boldsymbol{\omega} = \boldsymbol{\omega} - [BR]\boldsymbol{\omega}_r \tag{46}$$

The error in body angular acceleration is given by

$$\delta\dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}} - [BR]\dot{\boldsymbol{\omega}}_r + [\tilde{\omega}][BR]\boldsymbol{\omega}_r \tag{47}$$

Let the Lyapunov function be defined as[10,11,14]

$$U_T = \tfrac{1}{2}\delta\boldsymbol{\omega}^T I \delta\boldsymbol{\omega} + 2K\log(1 + \boldsymbol{\sigma}^T\boldsymbol{\sigma}) \tag{48}$$

where K is a scalar attitude feedback gain. Using the log function on a positive measure of tracking error in U_T results in the remarkable truth that $\boldsymbol{\sigma}$ appears linearly, without approximation, in $\dot{U}_T$ (Ref. 11):

$$\dot{U}_T = \delta\boldsymbol{\omega}^T I \delta\dot{\boldsymbol{\omega}} + \delta\boldsymbol{\omega}^T(K\boldsymbol{\sigma}) \tag{49}$$

After using the expression for the body angular acceleration and substituting the equations of motion given earlier, $\dot{U}_T$ is reduced to

$$\dot{U}_T = \delta\boldsymbol{\omega}^T(-[\tilde{\omega}]I\boldsymbol{\omega} + \boldsymbol{u} - I[BR]\dot{\boldsymbol{\omega}}_r + I[\tilde{\omega}][BR]\boldsymbol{\omega}_r + K\boldsymbol{\sigma}) \tag{50}$$

Note that, because we are using a trajectory rather than a fixed point as the energy reference, we cannot write $\dot{U}_T$ immediately using Eq. (8). Let us define the unsaturated control torque $\boldsymbol{u}_{us}$ as

$$\boldsymbol{u}_{us} = I([BR]\dot{\boldsymbol{\omega}}_r - [\tilde{\omega}][BR]\boldsymbol{\omega}_r) + [\tilde{\omega}]I\boldsymbol{\omega} - K\boldsymbol{\sigma} - P\delta\boldsymbol{\omega} \quad (51)$$

where P is a positive definite body angular velocity feedback gain matrix. This unsaturated control law reduces $\dot{U}_T$ of Eq. (50) to the nonpositive quadratic form

$$\dot{U}_T = -\delta\boldsymbol{\omega}^T P\delta\boldsymbol{\omega} \quad (52)$$

and causes the closed-loop equations of motion to be the elegantly simple linear form

$$I\delta\dot{\boldsymbol{\omega}} = -K\boldsymbol{\sigma} - P\delta\boldsymbol{\omega} \quad (53)$$

Because $\dot{U}_T$ is simply negative semidefinite, only stability and not asymptotic stability can be concluded. To prove that this unsaturated control law indeed leads to asymptotic stability, the higher-order derivatives of U_T need to be investigated. All points where $\dot{U}_T$ vanishes lie on the set Z where $\delta\boldsymbol{\omega} = 0$. The first nonzero higher-order derivative of U_T must be of odd order and negative definite for the system to be asymptotically stable.[9] The second derivative of U_T is

$$\frac{d^2 U_T}{dt^2} = -\delta\dot{\boldsymbol{\omega}}^T P\delta\boldsymbol{\omega} - \delta\boldsymbol{\omega}^T P\delta\dot{\boldsymbol{\omega}} \quad (54)$$

which is zero on Z where $\delta\boldsymbol{\omega} = 0$. The third derivative of U_T is

$$\frac{d^3 U_T}{dt^3} = -\delta\ddot{\boldsymbol{\omega}}^T P\delta\boldsymbol{\omega} - 2\delta\dot{\boldsymbol{\omega}}^T P\delta\dot{\boldsymbol{\omega}} - \delta\boldsymbol{\omega}^T P\delta\ddot{\boldsymbol{\omega}} < 0 \quad (55)$$

After using Eq. (53), evaluating d^3U_T/dt^3 is

$$\left.\frac{d^3 U_T}{dt^3}\right|_{\delta\omega=0} = -2(I^{-1}K\sigma)^T P(I^{-1}K\sigma) \quad (56)$$

which is negative definite on the set Z where $\delta\boldsymbol{\omega} = 0$. Therefore $\boldsymbol{u}_{us}$ is an asymptotically stabilizing tracking control law in the absence of model errors.

Now assume that the available control torque about the ith body axis is limited by $\boldsymbol{u}_{\max_i}$. Then following earlier analyses, we define a modified control law $\boldsymbol{u}$ as

$$\boldsymbol{u}_i = \begin{cases} \boldsymbol{u}_{us_i} & \text{for} \quad |\boldsymbol{u}_{us_i}| \le \boldsymbol{u}_{\max_i} \\ \boldsymbol{u}_{\max_i} \cdot \operatorname{sgn}(\boldsymbol{u}_{us_i}) & \text{for} \quad |\boldsymbol{u}_{us_i}| > \boldsymbol{u}_{\max_i} \end{cases} \quad (57)$$

A conservative stability boundary (a sufficient condition for stability) for this modified control torque is found to be

$$|(I([BR]\dot{\boldsymbol{\omega}}_r - [\tilde{\omega}][BR]\boldsymbol{\omega}_r) + [\tilde{\omega}]I\boldsymbol{\omega} - K\boldsymbol{\sigma})_i| \le |\boldsymbol{u}_{\max_i}| \quad (58)$$

Note that, for this higher-dimensional system, this stability constraint may be overly conservative. The condition in Eq. (58) is violated if the inequality fails about any one axis. As will be shown in the following example, for higher-dimensional cases the region where the stability inequality constraint is violated will typically be larger than the region of positive $\dot{U}_T$.

If zero reference motion is assumed from the beginning, then the preceding analysis leads to a globally asymptotically stable regulator for bounded torques. Thus, if $\dot{x}_r(t) \to 0$ at some final time t_f, then thereafter the end game, i.e., for $t > t_f$, has global asymptotic stability. Following this analysis, the unsaturated control torque becomes the simple linear control law

$$\boldsymbol{u}_{us} = -K\boldsymbol{\sigma} - P\delta\boldsymbol{\omega} \quad (59)$$

and the modified control torque is the same as before in Eq. (57). Note that this control law guarantees to return a rigid body from any arbitrarily large errors in orientation and angular velocity to a specified orientation at rest, assuming of course that enough fuel is available. Thus, global nonlinear controllability and stability are guaranteed. By using the modified control torque, reaching a $\boldsymbol{u}_{\max_i}$ about one of the body axes does not affect whether $\dot{U}_T$ is negative and, thus, does not affect the asymptotic stability. Because the reference motion and model nonlinearity affects the structure of the unsaturated control law in Eq. (51) and the stability boundary in Eq. (58), the robustness of this approach to tracking controller design requires further study for each family of maneuvers and estimates of model uncertainty.

The control law in Eq. (57) is illustrated with the following numerical simulation. The diagonal inertia matrix has the entries 385, 298, and 212 kgm^2. The reference maneuver is a smoothed near-minimum-time maneuver[14] starting at rest from the 3-1-3 Euler angles (−20, 15, and 4 deg) to the angles (40, 35, and 40 deg) with a final body angular velocity of (0, 1, and 0 deg/s). This type of open-loop reference maneuver replaces any instantaneous torque switches with cubically splined ones. The final maneuver time is 112 s. The initial attitude error in 3-1-3 Euler angles is (1, −2, and 1 deg). The initial body angular velocity error is (−0.025, 0.1, and 0.025 deg/s). The $\boldsymbol{u}_{\max}$ vector containing the maximum available torque about each body axis is (0.15, 0.2, and 0.15 Nm). The simulation was purposely chosen to periodically saturate the controls and violate the stability constraint of Eq. (58).

As with the numerical simulation in example 3, the time regions where Eq. (58) is violated due to torque saturation are shaded a light gray in the following figures. As a comparison, the regions where $\dot{U}_T$ is actually positive are shaded a dark gray. As expected, the stability constraint violations of Eq. (58) are overly conservative and no longer coincide with areas of positive $\dot{U}_T$ as they did with the earlier one-dimensional example. The maneuver is behaving in a stable fashion despite having large regions with stability constraint violations. Note that $\dot{U}_T$ is ultimately strictly negative after $t = 80$ s. Because Eq. (58) is very restrictive, the reference maneuvers for such torque saturated rotations often need to be designed interactively using Eq. (58) as an initial estimate.

The attitude tracking error $\boldsymbol{\sigma}$ is shown in Fig. 4. The initial tracking error is reduced to almost zero by the maneuver end. This occurs despite the stability boundary being violated during two time spans. Keep in mind that having occasional excursions of $\dot{U}_T > 0$ does not guarantee instability; it simply cannot guarantee stability.

The angular velocity tracking error $\delta\boldsymbol{\omega}$ is shown in Fig. 5. It too is reduced to near zero by maneuver end. The torque vector $\boldsymbol{u}$ is shown in Fig. 6. As in Fig. 2, the regions where $\dot{U}_T$ is positive are a subset of the regions where the torque about one of the body axes is saturated. Note also that not always are all three body axes

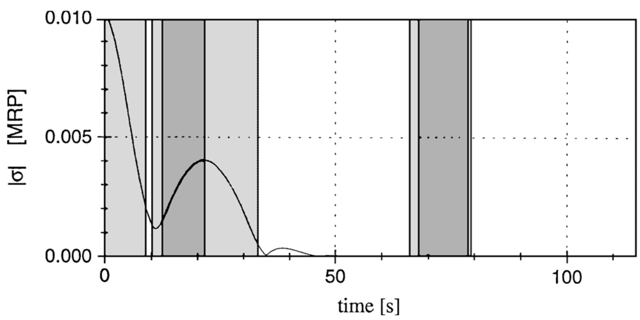

Fig. 4 Attitude tracking error norm.

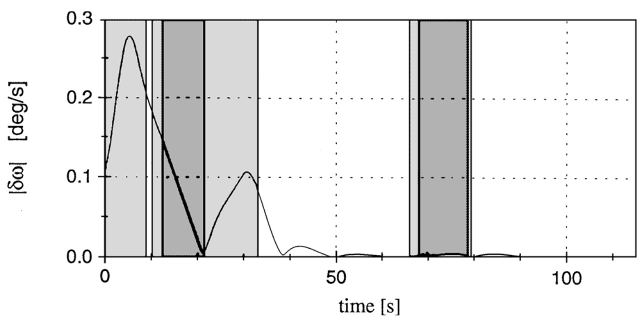

Fig. 5 Angular velocity tracking error norm.

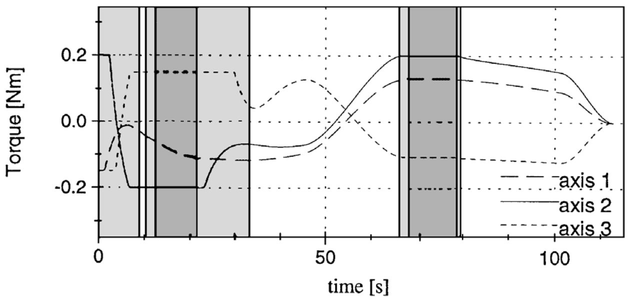

Fig. 6 Control torque vector.

saturated. The simulation contains cases where only one or two body axes are saturated. Note that the light gray stability constraint region of Eq. (58) extends over most of the torque saturated regions and even over some unsaturated regions. However, the dark gray regions of positive $\dot{U}_T$ only actually cover a smaller portion of the saturated torque cases. Covering some unsaturated regions further illustrates the conservative nature of the stability constraint in Eq. (58) because all unsaturated regions were shown to be asymptotically stable.

Concluding Remarks

A study is presented of Lyapunov optimality and the role of saturation constraints when using Lyapunov's direct method to design nonlinear feedback controllers for mechanical systems. For the special and usual case of controlled dynamics near a fixed point, it is shown how to efficiently design control laws and analyze global closed-loop stability properties. These laws are robust with respect to dynamical model errors because they are derived from a kinematic work/energy principle. For the case of tracking-type controllers, the fixed-point developments are extended and it is shown how to design controllers and analyze closed-loop stability using Lyapunov's direct method. Certain difficulties and pitfalls are noted due to saturation constraints, conservativeness of stability sufficient condition, and robustness issues.

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